

# Clique Coverings and Claw-free Graphs

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## Abstract

Let  $\mathcal{C}$  be a clique covering for  $E(G)$  and let  $v$  be a vertex of  $G$ . The valency of vertex  $v$  (with respect to  $\mathcal{C}$ ), denoted by  $val_{\mathcal{C}}(v)$ , is the number of cliques in  $\mathcal{C}$  containing  $v$ . The local clique cover number of  $G$ , denoted by  $lcc(G)$ , is defined as the smallest integer  $k$ , for which there exists a clique covering for  $E(G)$  such that  $val_{\mathcal{C}}(v)$  is at most  $k$ , for every vertex  $v \in V(G)$ . In this paper, among other results, we prove that if  $G$  is a claw-free graph, then  $lcc(G) + \chi(G) \leq n + 1$ .

**Keywords:** edge clique covering; local clique covering; chromatic number; claw-free graph; sigma clique partition; Nordhaus-Gaddum inequality.

**MSC:** 05C70

## 1 Introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph  $G$ , we mean a subset of mutually adjacent vertices of  $G$  as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices. A *clique covering* for  $E(G)$  is defined as a family of cliques of  $G$  such that every edge of  $G$  lies in at least one of the cliques comprising this family.

Let  $\mathcal{C}$  be a clique covering for  $E(G)$  and let  $v$  be a vertex of  $G$ . *Valency* of vertex  $v$  (with respect to  $\mathcal{C}$ ), denoted by  $val_{\mathcal{C}}(v)$ , is defined to be the number of cliques in  $\mathcal{C}$  containing  $v$ . A number of different variants of the clique cover number have been investigated in the literature. The *local clique cover number* of  $G$ , denoted by  $lcc(G)$ , is defined as the smallest

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integer  $k$ , for which there exists a clique covering for  $G$  such that  $val_C(v)$  is at most  $k$ , for every vertex  $v \in V(G)$ .

This parameter may be interpreted as a variety of different invariants of the graph and the problem relates to some well-known problems such as line graphs of hypergraphs, intersection representation and Kneser representation of graphs. For example,  $lcc(G)$  is the minimum integer  $k$  such that  $G$  is the line graph of a  $k$ -uniform hypergraph. By this interpretation,  $lcc(G) \leq 2$  if and only if  $G$  is the line graph of a multigraph.

There is a characterization by a list of seven forbidden induced subgraphs and a polynomial-time algorithm for the recognition that  $G$  is the line graph of a multigraph [3, 15]. On the other hand, L. Lovász in [16] proved that there is no characterization by a finite list of forbidden induced subgraphs for the graphs which are line graphs of some 3-uniform hypergraphs. Moreover, it was proved that the decision problem whether  $G$  is the line graph of a  $k$ -uniform hypergraph, for fixed  $k \geq 4$ , and the problem of recognizing line graphs of 3-uniform hypergraphs without multiple edges are NP-complete [18].

For a vertex  $v \in V(G)$ , its *(open) neighborhood*  $N(v)$  is the set of all neighbors of  $v$  in  $G$ , while its *closed neighborhood*  $N[v]$  is defined as  $N[v] := N(v) \cup \{v\}$ . Moreover, let  $\overline{G}$  stand for the complement of  $G$ , and let  $\Delta(G)$  and  $\delta(G)$  be the maximum degree and the minimum degree of  $G$ , respectively. The subgraph induced by a set  $Y \subset V(G)$  will be denoted by  $G[Y]$ . By the notations of  $\alpha(G)$ ,  $\omega(G)$ , and  $\chi(G)$  we mean the independence number, clique number, and chromatic number of  $G$ , respectively.

In 1956 E. A. Nordhaus and J. W. Gaddum proved the following theorem for the chromatic number of a graph  $G$  and its complement,  $\overline{G}$ .

**Theorem 1.** [17] *Let  $G$  be a graph on  $n$  vertices. Then  $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$ .*

Later on, similar results for other graph parameters have been found which are known as Nordhaus-Gaddum type theorems. In the literature there are several hundred papers considering inequalities of this type for many other graph invariants. For a survey of Nordhaus-Gaddum type estimates see [1].

In this paper, we consider the following two conjectures on local clique cover number.

**Conjecture 2.** For every graph  $G$  on  $n$  vertices,

$$lcc(G) + lcc(\overline{G}) \leq n. \quad (1)$$

This conjecture proposed by R. Javadi, Z. Maleki and B. Omoomi in 2012. Note that Conjecture 2 is a Nordhaus-Gaddum type inequality concerning the local clique cover number of  $G$ .

The second author with R. Javadi and B. Omoomi suggested the following weakening of Conjecture 2.

**Conjecture 3.** For every graph  $G$  on  $n$  vertices,

$$\text{lcc}(G) + \chi(G) \leq n + 1. \quad (2)$$

Let  $G_1$  and  $G_2$  be graphs with disjoint vertex sets  $V(G_1)$  and  $V(G_2)$  and edge sets  $E(G_1)$  and  $E(G_2)$ . The disjoint union of  $G_1$  and  $G_2$ , denoted by  $G_1 \dot{\cup} G_2$ , is the graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2)$ .

**Lemma 4.** *Let  $\mathcal{G}$  be a family of graphs which is closed under the operation of taking disjoint union with an isolated vertex. If Conjecture 2 is true for every  $G \in \mathcal{G}$ , then Conjecture 3 is also true for every  $G \in \mathcal{G}$ .*

*Proof.* Let  $G \in \mathcal{G}$  and consider the disjoint union  $H = G \dot{\cup} \{v\}$ . Observe that  $\text{lcc}(G) = \text{lcc}(H)$ . Hence, assuming that each member of  $\mathcal{G}$  satisfies Conjecture 2, we have  $\text{lcc}(G) + \text{lcc}(\overline{H}) \leq |V(H)|$ . Now, fix a clique covering  $\mathcal{C}$  for  $\overline{H}$ . Clearly,  $\chi(G) \leq \text{val}_{\mathcal{C}}(v) \leq \text{lcc}(\overline{H})$ . These two inequalities together imply  $\text{lcc}(G) = \text{lcc}(H) \leq |V(H)| - \text{lcc}(\overline{H}) \leq |V(G)| + 1 - \chi(G)$ .  $\square$

## 2 Proof of some variants of the conjectures

Let  $k$  be an integer and let  $G$  be a graph such that  $k \leq \deg(x) \leq k + 1$ , for every vertex  $x \in V(G)$ . Then  $\text{lcc}(G) \leq k + 1$  and  $\text{lcc}(\overline{G}) \leq n - 1 - k$ . Thus, inequality (1) holds for  $G$ . Also, If  $G$  is a triangle-free graph, then for a vertex  $v$  which has the maximum degree in  $G$ ,  $N(v)$  can properly be colored by one color. Thus,  $\chi(G) \leq n + 1 - \Delta(G)$ . Since  $\text{lcc}(G) = \Delta(G)$ , Conjecture 3 is true for triangle-free graphs. In what follows we prove that not only (2) but also (1) holds if  $\overline{G}$  is triangle-free.

**Theorem 5.** *Let  $G$  be a graph on  $n$  vertices. If  $\alpha(G) = 2$ , then  $\text{lcc}(G) + \text{lcc}(\overline{G}) \leq n$ .*

*Proof.* Clearly,  $\text{lcc}(\overline{G}) \leq \Delta(\overline{G}) = n - 1 - \delta(G)$ . It is enough to show that  $\text{lcc}(G) \leq \delta(G) + 1$ . Let  $v$  be a vertex of minimum degree in  $G$ , and let  $K \subset V(G)$  be the set of vertices which are not adjacent to  $v$ . Since  $\alpha(G) = 2$ , the induced subgraph on  $K$ ,  $G[K]$ , is a clique in  $G$ . Now, for every vertex  $u_i \in N(v)$ , let  $C_i := (N(u_i) \cap K) \cup \{u_i\}$  and define  $C_{\delta(G)+1} := G[K]$ . These cliques along with the collection of those edges which are not covered by the cliques  $C_1, \dots, C_{\delta(G)+1}$  comprise a clique covering for  $G$ , say  $\mathcal{C}$ . It can be checked easily that  $\text{val}_{\mathcal{C}}(v) = \delta(G)$  and  $\text{val}(x) \leq \delta(G) + 1$ , for every vertex  $x \in V(G) - v$ .  $\square$

It is well-known that  $\frac{n}{\alpha(G)}$  and  $\omega(G)$  are lower bounds for  $\chi(G)$ , the chromatic number of  $G$ . We show that, if we replace  $\chi(G)$  with any of these two general lower bounds in Conjecture 3, then the inequality holds.

**Proposition 6.** *Let  $G$  be a graph with  $n$  vertices. Then  $\text{lcc}(G) + \omega(G) \leq n + 1$ .*

*Proof.* Assume that  $K \subset V(G)$  is a clique of size  $\omega$ . For every vertex  $v_i \in V(G) - K$ ,  $1 \leq i \leq n - \omega$ , define  $C_i := (N(v_i) \cap K) \cup \{v_i\}$ , and let  $C_{n-\omega+1} := G[K]$ . Now, let  $F$  be the set of all the edges which are not covered by the cliques  $C_1, \dots, C_{n-\omega+1}$ . Clearly, the cliques  $C_i$  for  $1 \leq i \leq n - \omega + 1$  together with  $F$  form a clique covering  $\mathcal{C}$  for  $G$ . If  $x \in K$ , then  $\text{val}_{\mathcal{C}}(x) \leq 1 + n - \omega(G)$ , and for vertex  $v_i \in V(G) - K$ ,  $\text{val}_{\mathcal{C}}(v_i) \leq n - \omega(G)$ .  $\square$

Before proving the other inequality  $\text{lcc}(G) + \frac{n}{\alpha(G)} \leq n + 1$ , we verify a stronger statement involving local parameters. Let  $\alpha_G(v) = \alpha(G[N(v)])$  be the maximum number of independent vertices in the neighborhood of vertex  $v$ , and let the *local independence number* of graph  $G$  be defined as  $\alpha_L(G) = \max_{v \in V(G)} \alpha_G(v)$ . Clearly,  $\alpha_G(v) \leq \alpha_L(G) \leq \alpha(G)$ . Further,  $\alpha_G(v) \geq 1$  holds if and only if  $v$  has at least one neighbor, while  $\alpha_G(v) \leq 1$  is equivalent to that the *closed neighborhood*  $N_G[v] = N(v) \cup \{v\}$  induces a clique.

**Theorem 7.** *For every graph  $G$  of order  $n$ , there exists a clique covering  $\mathcal{C}$  such that for each non-isolated vertex  $v \in V(G)$  the inequality  $\text{val}_{\mathcal{C}}(v) + \frac{n}{\alpha_G(v)} \leq n + 1$  holds.*

*Proof.* A clique covering will be called *good* if it satisfies the requirement given in the theorem. Since the statement is true for all graphs of order  $n \leq 3$ , we may proceed by induction on  $n$ . Let  $x$  and  $y$  be two adjacent vertices of  $G$ . By the induction hypothesis, there is a good clique covering,  $\mathcal{C}'$ , for  $G' = G - \{x, y\}$ . We introduce the notations  $N_1 := N(x) - N[y]$ ,  $N_2 := N(y) - N[x]$ , and  $N_{1,2} := N(x) \cap N(y)$ . To obtain a good clique covering  $\mathcal{C}$  of  $G$  from  $\mathcal{C}'$ , we perform the following steps.

1. To handle vertices whose neighbors are completely adjacent, observe that every vertex  $u$  from  $N_1 \cup N_2 \cup N_{1,2}$  with  $\alpha_G(u) = 1$  and  $\deg_{G'}(u) \geq 1$  has  $\alpha_{G'}(u) = 1$  and hence it is covered by the clique  $N_{G'}[u]$  in the good covering  $\mathcal{C}'$ . Now, for each such vertex  $u$ ,  $N_{G'}[u]$  is extended by  $x$ , by  $y$  or by both  $x$  and  $y$  respectively, if  $u \in N_1$ ,  $u \in N_2$  or  $u \in N_{1,2}$ .
2. If  $\alpha_G(x) = 1 < \alpha_G(y)$ , take the clique  $N_G[x]$ ; if  $\alpha_G(y) = 1 < \alpha_G(x)$ , take the clique  $N_G[y]$ ; and if  $\alpha_G(x) = \alpha_G(y) = 1$ , take the clique  $N_G[x] = N_G[y]$  into the covering  $\mathcal{C}$  (if they were not included in step (1)).
3. If there still exist some uncovered edges between  $x$  and  $N_1$ , we consider the set  $N'_1 = \{v \in N_1 \mid xv \text{ is uncovered}\}$  and partition it into some number of adjacent vertex pairs (inducing independent edges) and at most  $\alpha(G(N'_1))$  isolated vertices. Then, we extend each of them with  $x$  to a  $K_3$  or  $K_2$ , and insert these cliques into the covering  $\mathcal{C}$ . This way, we get at most  $\frac{|N'_1| - \alpha(G(N'_1))}{2} + \alpha(G(N'_1))$  new cliques. Then, we define  $N'_2$  and  $N'_{1,2}$  analogously, and do the corresponding partitioning procedure for  $N'_2$  and  $N'_{1,2}$ , extending every part of those partitions with  $y$  or with  $\{x, y\}$ , respectively.
4. If the edge  $xy$  remained uncovered, we take it as a clique into the covering  $\mathcal{C}$ .

It is easy to check that  $\mathcal{C}$  is a clique covering in  $G$ . We prove that it is good.

First note that after performing Step 1, each vertex  $v \in V(G) - \{x, y\}$  has the same valency as in  $\mathcal{C}'$ . Moreover, if two adjacent vertices, say  $u$  and  $x$ , have  $\alpha_G(u) = \alpha_G(x) = 1$ , then  $N_G[u] = N_G[x]$  must hold. Hence, if  $u \in V(G) - \{x, y\}$  and  $\alpha_G(u) = 1$ , then  $u$  is incident with only one clique from  $\mathcal{C}$ . Thus,  $val_{\mathcal{C}}(u) + \frac{n}{\alpha_G(u)} = 1 + n$ . If  $v$  is a vertex from  $V(G) - \{x, y\}$  and  $\alpha_G(v) \geq 2$ , then the valency of  $v$  might increase in Step 2 or 3, but not in both. Therefore,  $val_{\mathcal{C}}(v) \leq val_{\mathcal{C}'}(v) + 1$ , and clearly  $\alpha_{G'}(v) \leq \alpha_G(v)$ . Since  $\mathcal{C}'$  is assumed to be good, these facts together imply

$$val_{\mathcal{C}}(v) + \frac{n}{\alpha_G(v)} \leq val_{\mathcal{C}'}(v) + 1 + \frac{n-2}{\alpha_{G'}(v)} + \frac{2}{\alpha_G(v)} \leq n+1.$$

Now, consider the vertex  $x$ . If  $\alpha_G(x) = 1$ , it is covered by only one clique (induced by its closed neighborhood), which was added to  $\mathcal{C}$  in Step 1 or 2. In this case  $val_{\mathcal{C}}(x) + \frac{n}{\alpha_G(x)} = n+1$ . Also if  $\alpha_G(x) \geq \frac{n}{2}$ , the trivial bound  $val_{\mathcal{C}}(x) \leq \deg(x) \leq n-1$  implies the desired inequality. Hence, we may suppose  $2 \leq \alpha_G(x) < \frac{n}{2}$ .

Let us denote by  $s$  the number of cliques covering  $x$  which were added to  $\mathcal{C}$  in Step 1. Choose one vertex  $u_i$  with  $\alpha_G(u_i) = 1$  from each of these  $s$  cliques. The closed neighborhoods  $N[u_i]$  are pairwise different cliques. Thus, the obtained vertex set  $S$  is independent. By the definitions of  $N'_1$  and  $N'_{1,2}$ , there exist no edges between  $S$  and  $N'_1 \cup N'_{1,2}$ . Thus,  $\alpha(G(N'_1)) \leq \alpha_G(x) - s$  and  $\alpha(G(N'_{1,2})) \leq \alpha_G(x) - s$ . Also,  $|N'_1| + |N'_{1,2}| \leq |N_1| + |N_{1,2}| - s = \deg(x) - 1 - s$  follows.

- If  $N_{1,2} \neq \emptyset$  and  $\alpha_G(y) > 1$ , then

$$\begin{aligned} val_{\mathcal{C}}(x) &\leq \frac{|N'_1| - \alpha(G(N'_1))}{2} + \alpha(G(N'_1)) + \frac{|N'_{1,2}| - \alpha(G(N'_{1,2}))}{2} + \alpha(G(N'_{1,2})) + s \\ &= \frac{|N'_1| + |N'_{1,2}|}{2} + \frac{\alpha(G(N'_1)) + \alpha(G(N'_{1,2}))}{2} + s \\ &\leq \frac{\deg(x) - 1 - s}{2} + \frac{2\alpha_G(x) - 2s}{2} + s \leq \frac{n-2}{2} + \alpha_G(x). \end{aligned}$$

On the other hand, our assumption  $2 \leq \alpha_G(x) < \frac{n}{2}$  implies that  $\alpha_G(x) + \frac{n}{\alpha_G(x)} \leq 2 + \frac{n}{2}$ . Thus,

$$val_{\mathcal{C}}(x) + \frac{n}{\alpha_G(x)} \leq \frac{n-2}{2} + \alpha_G(x) + \frac{n}{\alpha_G(x)} \leq \frac{n-2}{2} + 2 + \frac{n}{2} = n+1.$$

- If  $N_{1,2} \neq \emptyset$  and  $\alpha_G(y) = 1$ , all edges between  $N_{1,2}$  and  $x$  are covered by the clique  $N_G[y]$ , which was added to  $\mathcal{C}$  in Step 2 (or maybe earlier, in Step 1). Hence,  $N'_{1,2} = \emptyset$

and we have

$$\begin{aligned}
val_{\mathcal{C}}(x) &\leq \frac{|N'_1| - \alpha(G(N'_1))}{2} + \alpha(G(N'_1)) + 1 + s \\
&= \frac{|N'_1|}{2} + \frac{\alpha(G(N'_1))}{2} + 1 + s \\
&\leq \frac{\deg(x) - 1 - s}{2} + \frac{\alpha_G(x) - s}{2} + 1 + s \leq \frac{n-2}{2} + \alpha_G(x).
\end{aligned}$$

Again, we may conclude  $val_{\mathcal{C}}(x) + \frac{n}{\alpha_G(x)} \leq n + 1$ .

- If  $N_{1,2} = \emptyset$ , the clique  $xy$  was added to  $\mathcal{C}$  in Step 4, and the same estimation holds as in the previous case.

One can show similarly that  $val_{\mathcal{C}}(y) + \frac{n}{\alpha_G(y)} \leq n + 1$ . This completes the proof.  $\square$

Since for every  $v \in V(G)$ ,  $\alpha_G(v) \leq \alpha_L(G) \leq \alpha(G)$ , we have the following immediate consequences.

**Corollary 8.** *Let  $G$  be a graph of order  $n$ . Then*

- (i)  $\text{lcc}(G) + \frac{n}{\alpha_L(G)} \leq n + 1$ ;
- (ii)  $\text{lcc}(G) + \frac{n}{\alpha(G)} \leq n + 1$ .

On the other hand,  $val_{\mathcal{C}}(v) \geq \alpha_G(v)$ , for every arbitrary clique covering  $\mathcal{C}$ . Hence,  $\text{lcc}(G) \geq \alpha_L(G)$ . (But  $\text{lcc}(G) < \alpha(G)$  may be true.) Also, it is easy to see that  $\text{lcc}(G) \geq \frac{\Delta(G)}{\omega-1}$ . Next we observe that replacing  $\text{lcc}(G)$  with  $\alpha(G)$  or  $\frac{\Delta(G)}{\omega-1}$  in Conjecture 3, valid inequalities are obtained.

**Proposition 9.** *If  $G$  is a graph on  $n$  vertices, then*

1.  $\frac{\Delta(G)}{\omega-1} + \chi(G) \leq n + 1$ , and equality holds if and only if  $G$  is the complete graph  $K_n$  or the star  $K_{1,n-1}$ ;
2.  $\alpha(G) + \chi(G) \leq n + 1$ , and equality holds if and only if there exists a vertex  $v \in V(G)$  such that  $N(v)$  induces a complete graph and  $V(G) \setminus N(v)$  is an independent set.

*Proof.* To prove (1), first note that it is showed in [10] that there are only two types of graphs  $G$  for which  $\chi(G) + \chi(\bar{G}) = n + 1$ ,

- (a) if  $V(G) = K \cup S$  where  $K$  is a clique and  $S$  is an independent set, sharing a vertex  $K \cap S = \{u\}$ , or

(b)  $G$  is obtained from (a) by substituting  $C_5$  into  $u$ .

Now, we estimate  $\frac{\Delta(G)}{\omega-1} + \chi(G)$  as follows. We write  $\theta$  for the clique covering number (minimum number of complete subgraphs whose union is the entire vertex set, that is the chromatic number of the complementary graph). Let  $x$  be a vertex of degree  $\Delta = \Delta(G)$ . We have

$$\frac{\Delta}{\omega-1} \leq \theta(G[N(x)]) \leq \theta(G) \leq n+1 - \chi(G),$$

where the last inequality is the Nordhaus-Gaddum theorem (Theorem 1). Thus, in order to have  $\frac{\Delta}{\omega-1} + \chi = n+1$ , it is necessary that  $G$  is of type (a) or (b). We shall see that (b) is not good enough, and (a) yields  $G = K_n$  or  $G = K_{1,n-1}$ .

Note that equality does not hold for  $G = C_5$ , therefore in (b) we have  $k > 0$ . Let  $|K - u| = k$  and  $|S - u| = s$  in (a). Then after substitution of  $C_5$ , we have  $n = k + s + 5$ ,  $\Delta \leq n - 1$ ,  $\omega = k + 2$  (with  $k > 0$ ), and  $\chi = k + 3$ . Therefore, the most favorable case is  $s = 0$ , because increasing  $s$  by 1 makes  $n+1$  increase by 1, while the left-hand side of the inequality increases by at most  $1/2$ . Hence, in the best case we have  $n = k + 5 \geq 6$ , and

$$\frac{\Delta}{\omega-1} + \chi = \frac{n-1}{n-3} + n - 2 < n + 1$$

Now, we consider case (a). Here, again we have  $k > 0$  and  $\Delta \leq n - 1$ , moreover now  $n = k + s + 1$ ,  $\omega = k + 1$ , and  $\chi = k + 1$ . Thus

$$\frac{\Delta}{\omega-1} + \chi \leq \frac{(k+s)}{k} + k + 1 \leq k + s + 2$$

with equality if and only if  $s/k = s$ , that is  $k = 1$  or  $s = 0$ , where for the case  $k = 1$  we also have to ensure  $\Delta = s + 1$ . This completes the proof of (1).

To see (2), consider an independent set  $A$  of cardinality  $\alpha = \alpha(G)$ . A proper  $(n - \alpha + 1)$ -coloring always exists as we can assign color 1 to all vertices from  $A$  and the further  $n - \alpha$  vertices are assigned with pairwise different colors. Hence,  $\chi(G) \leq n - \alpha + 1$  holds for every graph. Moreover, if the graph induced by  $V(G) \setminus A$  is not complete, we can color it properly by using fewer than  $n - \alpha$  colors that yields a proper coloring of  $G$  with fewer than  $n - \alpha + 1$  colors. Therefore,  $\chi(G) = n - \alpha + 1$  may hold only if  $V(G) \setminus A$  induces a complete graph. In this case,  $G$  is a split graph. Since split graphs are chordal and chordal graphs are perfect [8],  $\omega(G) = \chi(G) = n - \alpha + 1$ . Consequently, if (2) holds with equality, there exists a vertex  $v \in A$  which is adjacent to all vertices from  $V(G) \setminus A$ . This vertex fulfills our conditions as  $N(v)$  is a clique and  $V(G) \setminus N(v)$  is an independent set.

On the other hand, if a vertex  $v'$  with such a property exists in  $G$ , then the graph cannot be colored with fewer than  $|N(v')| + 1$  colors. This implies  $\chi = n - \alpha + 1$  and completes the proof of the second statement.  $\square$

### 3 Claw-free graphs

Several related problems (say, perfect graph conjecture, to mention just the most famous one) are easier for *claw-free graphs*, i.e. for graphs not containing  $K_{1,3}$  as an induced subgraph, other problems (say, complexity of finding chromatic number) are not. (For a survey of results on claw-free graphs see e.g. [9].) Concerning local clique cover number, R. Javadi et al. showed in [12] that if  $G$  is a claw-free graph then  $\text{lcc}(G) \leq c \frac{\Delta(G)}{\log(\Delta(G))}$ , for a constant  $c$ . In this section, we are going to prove that Conjecture 3 does hold for claw-free graphs.

To prove the main result of this section, we use the following definition and theorem of Balogh et al. [2].

**Definition 10.** [2] A graph  $G$  is  $(s, t)$ -splittable if  $V(G)$  can be partitioned into two sets  $S$  and  $T$  such that  $\chi(G[S]) \geq s$  and  $\chi(G[T]) \geq t$ . For  $2 \leq s \leq \chi(G) - 1$ , we say that  $G$  is  $s$ -splittable if  $G$  is  $(s, \chi(G) - s + 1)$ -splittable.

**Theorem 11.** [2] Let  $s \geq 2$  be an integer. Let  $G$  be a graph with  $\alpha(G) = 2$  and  $\chi(G) > \max\{\omega, s\}$ . Then  $G$  is  $s$ -splittable.

Now we prove:

**Theorem 12.** Let  $G$  be a claw-free graph with  $n$  vertices. Then  $\text{lcc}(G) + \chi(G) \leq n + 1$ . Moreover, for every  $n \geq 4$ , there exist several claw-free graphs with  $n$  vertices such that equality holds.

*Proof.* We prove the theorem by induction on  $n$ . For small values of  $n$ , it is easy to check that a claw-free graph with  $n$  vertices satisfies the inequality. Also, the assertion is obvious for  $\alpha(G) = 1$ .

Let  $G$  be a claw-free graph on  $n$  vertices. First, we consider the case that  $\alpha(G) \geq 3$ . Let  $T$  be an independent set of size three. By the induction hypothesis,  $G - T$  has a clique covering  $\mathcal{C}'$  such that every vertex  $x \in V(G - T)$  has

$$\text{val}_{\mathcal{C}'}(x) \leq (n - 3) + 1 - \chi(G - T) \leq n - 2 - (\chi(G) - 1) = n - 1 - \chi(G). \quad (3)$$

Now, for every vertex  $u \in T$ , partition  $N(u)$  into the  $\chi(\overline{G[N(u)]})$  vertex-disjoint cliques. Then, add vertex  $u$  to each clique to cover all the edges incident to  $u$ . These cliques along with cliques in an optimum clique covering of  $G - T$  form a clique covering, say  $\mathcal{C}$ , for  $G$ . Let  $u \in T$  and  $x \in G - T$ . Then we have

$$\begin{aligned} \text{val}_{\mathcal{C}}(u) &= \chi(\overline{G[N(u)]}) \leq \chi(\overline{G}) \leq n + 1 - \chi(G), \\ \text{val}_{\mathcal{C}}(x) &\leq \text{val}_{\mathcal{C}'}(x) + |N_G(x) \cap T|. \end{aligned}$$

Since  $G$  is claw free,  $|N_G(x) \cap T| \leq 2$ . Thus, by Inequality (3),  $\text{lcc}(G) \leq n + 1 - \chi(G)$ .



Consider now the case  $\alpha(G) = 2$ . By Proposition 6 we may assume that  $\chi(G) > \omega(G)$ . Moreover, as the statement clearly holds when  $\chi(G) \leq 2$ , we may also suppose that  $\chi(G) \geq 3$ . Then Theorem 11 with  $s = 2$  implies that  $V(G)$  can be partitioned into two parts, say  $A$  and  $B$ , such that  $\chi(G[A]) \geq 2$  and  $\chi(G[B]) \geq \chi(G) - 1$ . We assume, without loss of generality, that  $A = \{u_1, u_2\}$ , where the vertices  $u_1$  and  $u_2$  are adjacent. Then  $\chi(G - \{u_1, u_2\}) \geq \chi(G) - 1$ .

We will use the notation  $N_1 := N(u_1) - N[u_2]$ ,  $N_2 := N(u_2) - N[u_1]$ , and  $N_{1,2} := N(u_1) \cap N(u_2)$ . Since  $G$  is claw-free,  $N_i \cup \{u_i\}$  induces a clique for  $i = 1, 2$ . Starting with an optimal clique covering  $\mathcal{C}''$  for  $G - \{u_1, u_2\}$ , we will construct a clique covering  $\mathcal{C}$  for  $G$  such that  $\text{val}_{\mathcal{C}}(v) \leq n + 1 - \chi(G)$  holds for every vertex  $v$ .

If  $N_{1,2} = \emptyset$ , then  $\mathcal{C} := \mathcal{C}'' \cup \{N_1 \cup \{u_1\}, N_2 \cup \{u_2\}, \{u_1, u_2\}\}$  is a clique covering for  $G$ . We observe that  $\text{val}_{\mathcal{C}}(u_i) \leq 2$  holds for  $i = 1, 2$  and

$$\text{val}_{\mathcal{C}}(v) \leq \text{val}_{\mathcal{C}''}(v) + 1 \leq n - 1 - \chi(G - \{u_1, u_2\}) + 1 \leq n - \chi(G)$$

for each vertex  $v$  from  $V(G - \{u_1, u_2\})$ . Hence,  $\text{lcc}(G) \leq n + 1 - \chi(G)$ .

Otherwise, if  $N_{1,2} \neq \emptyset$ , partition  $N_{1,2}$  into at most  $\chi(\overline{G - \{u_1, u_2\}})$  cliques and extend each of them with the vertices  $u_1$  and  $u_2$ . These cliques together with  $N_1 \cup \{u_1\}$ ,  $N_2 \cup \{u_2\}$ , and with the cliques in  $\mathcal{C}''$  form a clique covering of  $G$ . We show that this clique covering  $\mathcal{C}$  satisfies  $\text{val}_{\mathcal{C}}(x) \leq n + 1 - \chi(G)$  for every vertex  $x \in V(G)$ . Note that  $\text{val}_{\mathcal{C}}(u_1) \leq \chi(\overline{G - \{u_1, u_2\}}) + 1$ , thus the Nordhaus-Gaddum inequality for chromatic number implies

$$\text{val}_{\mathcal{C}}(u_1) \leq (n - 2) + 1 - \chi(G - \{u_1, u_2\}) + 1 \leq n - \chi(G - \{u_1, u_2\}) \leq n + 1 - \chi(G).$$

Similarly, we have  $\text{val}_{\mathcal{C}}(u_2) \leq n + 1 - \chi(G)$ . For  $v \in V(G - \{u_1, u_2\})$ ,

$$\text{val}_{\mathcal{C}}(v) \leq \text{val}_{\mathcal{C}''}(v) + 1 \leq (n - 2) + 1 - \chi(G - \{u_1, u_2\}) + 1 \leq n - \chi(G) + 1.$$

Finally, we note that  $K_n$ ,  $K_n - K_2$ , and  $K_n - K_{1,2}$  are examples of claw-free graphs with  $n$  vertices such that  $\text{lcc}(G) + \chi(G) = n + 1$ .  $\square$

## 4 A Nordhaus-Gaddum type inequality

A *clique partition* of the edges of a graph  $G$  is a family of cliques such that every edge of  $G$  lies in exactly one member of the family. The *sigma clique partition number* of  $G$ ,  $\text{scp}(G)$ , is the smallest integer  $k$  for which there exists a clique partition of  $E(G)$  where the sum of the sizes of its cliques is at most  $k$ .

It was conjectured by G. O. H. Katona and T. Tarján, and proved in the papers [4, 13, 11], that for every graph  $G$  on  $n$  vertices,  $\text{scp}(G) \leq \lfloor n^2/2 \rfloor$  holds, with equality if and only if  $G$  is the complete bipartite graph  $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ .

Also, this parameter relates to a number of other well-known problems (see [6]). The second author and R. Javadi proved the following Nordhaus-Gaddum type theorem for  $\text{scp}$ .

**Theorem 13.** [5] *Let  $G$  be a graph with  $n$  vertices. Then*

$$\begin{aligned} \frac{31}{50}n^2 + O(n) &\leq \max\{\text{scp}(G) + \text{scp}(\overline{G})\} \leq \frac{9}{10}n^2 + O(n), \\ \frac{12}{125}n^4 + O(n^3) &< \max\{\text{scp}(G) \cdot \text{scp}(\overline{G})\} < \frac{81}{400}n^4 + O(n^3). \end{aligned}$$

In the following result we improve the upper bounds, from 0.9 to less than 0.77 and from 0.2025 to less than 0.15.

**Theorem 14.** *For every graph  $G$  with  $n$  vertices,*

$$\text{scp}(G) + \text{scp}(\overline{G}) \leq \frac{1203}{1568}n^2 + o(n^2) < 0.76722n^2 + o(n^2)$$

and

$$\text{scp}(G) \cdot \text{scp}(\overline{G}) \leq \frac{1447209}{9834496}n^4 + o(n^4) < 0.1471564n^4 + o(n^4).$$

*Proof.* Substantially improving on earlier estimates, P. Keevash and B. Sudakov [14] proved via a computer-aided calculation that every edge 2-coloring of  $K_n$  contains at least  $cn^2 - o(n^2)$  mutually edge-disjoint monochromatic triangles,<sup>1</sup> where

$$c = \frac{13}{196} + \frac{1}{84} - \frac{1}{1568} = \frac{365}{4704}.$$

In our context this means that we can select approximately  $cn^2$  triangles which together cover  $3cn^2$  edges in  $G$  and  $\overline{G}$  at the cost of  $3cn^2$ . The remaining edges will be viewed as copies of  $K_2$  in the clique partition to be constructed; they are counted with weight 2 in  $\text{scp}$ . In this way we obtain

$$\text{scp}(G) + \text{scp}(\overline{G}) \leq (1 - 3c)n^2 + o(n^2) = \frac{1203}{1568}n^2 + o(n^2).$$

This also implies the upper bound on  $\text{scp}(G) \cdot \text{scp}(\overline{G})$ . □

**Remark 15.** The smallest number of cliques in a clique partition of  $G$  is called the *clique partition number* of  $G$ . As a Nordhaus-Gaddum type inequality for parameter  $\text{cp}$ , D. de Caen et al. proved in [7] that

$$\begin{aligned} \text{cp}(G) + \text{cp}(\overline{G}) &\leq \frac{13}{30}n^2 - O(n) \approx 0.43333n^2 - O(n), \\ \text{cp}(G) \cdot \text{cp}(\overline{G}) &\leq \frac{169}{3600}n^4 + O(n^3) \approx 0.0469444n^4 + O(n^3). \end{aligned}$$

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<sup>1</sup>In the Abstract of [14] the authors announce the lower bound  $n^2/13$ , and in their Theorem 1.1 they state  $n^2/12.89$  (the rounded form of  $\frac{9}{116}n^2$ , but actually on p. 212 they prove the even better lower bound displayed above.)

Note that if it is possible to select some  $k$  edge-disjoint complete subgraphs in  $G$  and  $\overline{G}$  which together cover  $m$  edges, then  $\text{cp}(G) + \text{cp}(\overline{G}) \leq \binom{n}{2} + k - m$ . As observed within the proof of Theorem 14, the choices  $k = \frac{365}{4704}n^2 - o(n^2)$  and  $m = 3k$  are feasible for every  $G$  on  $n$  vertices, thus

$$\begin{aligned}\text{cp}(G) + \text{cp}(\overline{G}) &\leq \left(\frac{1}{2} - \frac{365}{2352}\right)n^2 + o(n^2) = \frac{811}{2352}n^2 + o(n^2) < 0.344813n^2 + o(n^2), \\ \text{cp}(G) \cdot \text{cp}(\overline{G}) &\leq \frac{657721}{22127616}n^4 + o(n^4) < 0.029724n^4 + o(n^4).\end{aligned}$$

These upper bounds improve the results of [7].

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